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Mathematical modelling and theoretical analysis of nonholonomic kinematic systems with a class of rheonomous affine constraints

Tatsuya Kai *

Department of Electrical Engineering, Faculty of Information Science and Electrical Engineering, Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan

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ABSTRACT

In this paper, we deal with kinematic control systems subject to a class of rheonomous affine constraints. We first define *A*-rheonomous affine constraints and explain a geometric representation method for them. Next, we derive a necessary and sufficient condition for complete nonholonomicity of the *A*-rheonomous affine constraints. Then, a mathematical model of nonholonomic kinematic systems with *A*-rheonomous affine constraints (NKSARAC), which is included in the class of nonlinear affine control systems, is introduced. Theoretical analysis on linearly-approximated systems and accessibility for the NKSARAC is also shown. Finally, we apply the results to some physical examples in order to confirm the effectiveness of them.

1. Introduction

A lot of researchers have studied nonlinear control systems subject to nonholonomic constraints, so-called *nonholonomic control systems* so far [1–4]. Nonholonomic systems are, in the simplest terms, defined as ones which are subject to nonintegrable constraints and whose behaviors must satisfy the constraints. There are a lot of examples of nonholonomic systems: mobile cars [5–7], trailers [8–10], space robots [11,12], acrobat robots [13,14], a rolling ball or coin on a plain [15–17], underactuated manipulators [18–20] and so on. Roughly speaking, researches on nonholonomic control systems can be classified into the two research fields: kinematic systems and dynamic systems. In both research fields, *linear constraints* which are linear in velocities have been mainly investigated. Kinematic systems are directly derived from nonholonomic constraints, and in particular linear constraints can be transformed into symmetrically affine control systems [6]. On the contrary, dynamic systems are derived from Euler–Lagrange equations with the constrained forces based on d'Alembert's principle [21,16]. There are two common characteristics between kinematic and dynamic systems: (i) Their linear approximated systems are uncontrollable. (ii) They are locally controllable, but not locally asymptotically stabilizable by any nonlinear smooth state feedback from Brockett's theorem [22]. Therefore, many control laws which avoid Brockett's condition have been proposed such as time-variant feedback, discontinuous feedback, and switching control laws.

However, there is another class of constraints which are affine in velocities and called *affine constraints*. It is a larger class of constraints than that of linear constraints. As shown in Fig. 1, a space robot with an initial angular momentum, a coin or a ball on a rotating table [17], a pneumatic tire [15], under-actuated manipulators and underwater vehicles [4] are typical examples of systems subject to affine constraints. Until now, there have been much less researches on affine constraints than those on linear constraints. So, we have focused on and studied affine constraints from the viewpoints of both mathematics

* Tel.: +81 92 802 3699; fax: +81 92 802 3705.
E-mail address: kai@ees.kyushu-u.ac.jp

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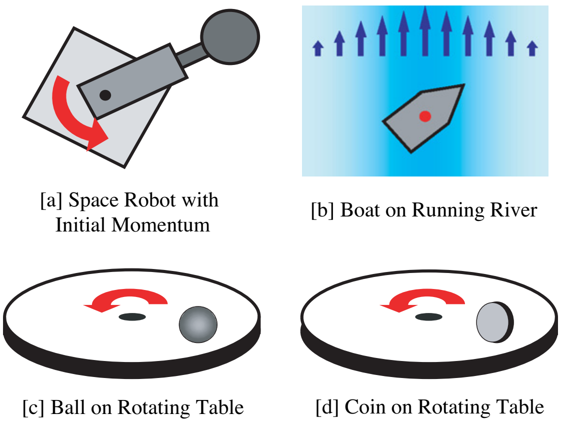

Fig. 1. Examples of systems subject to affine constraints.

and control theory [23–31]. Especially, in [23,24], we have derived the integrability and nonintegrability conditions for affine constraints, and investigated accessibility of nonholonomic kinematic systems with affine constraints (NKSAC) which are derived from affine constraints. Moreover, in [26] we have analyzed the NKSAC and shown the two interesting results: (i) There exists a class of systems whose linear approximations are controllable, and hence they are stabilizable by linear state feedback laws. (ii) There exists a class of systems such that Brockett's condition holds, i.e., they has a possibility of stabilization by nonlinear smooth state feedback laws. These facts are far beyond the well-known facts for nonholonomic systems subject to linear constraints.

All the constraints which are dealt with in the researches above do not contain the time variable, that is, *scleronomous constraints* [15]. However, there are nonholonomic mechanical systems whose constraints contain the time variable, for example, a coin or a ball on a rotating table at a time-varying angular velocity, a boat on a running river with a time-varying stream and so on (Fig. 1). These constraints are called *rheonomous* [15], and we need a new theory on rheonomous constraints in order to apply control theory to such systems.

Hence, the purpose of this paper is to analyze a class of rheonomous affine constraints (A-rheonomous affine constraints) based on nonlinear control theory and derive some fundamental characteristics of kinematic systems subject to them. This paper is organized as follows. In Section 2, we first define A-rheonomous affine constraints and introduce their geometric representation. Then, a necessary and sufficient condition on complete nonholonomicity of the rheonomous affine constraints is derived in terms of the rheonomous bracket which is a new operator for the geometric representation of the A-rheonomous affine constraints. In Section 3, we introduce *a nonholonomic kinematic system with A-rheonomous affine constraints (NKSARAC)* as a kinematic model, and investigate its linearly-approximated system and an accessibility condition for the NKSARAC. Finally, some physical examples are illustrated in order to check our new results in Section 4. Throughout this paper, manifolds, vector fields, one forms, functions and distributions are all assumed to be smooth.

2. Rheonomous affine constraints

2.1. Definition and geometric representation

In this subsection, we give a definition of rheonomous affine constraints that we consider throughout this paper. We denote the time variable by $t \in \mathbf{R}$ and a time interval by $I \subset \mathbf{R}$. Let Q be an n-dimensional configuration manifold and $q = [q_1 \cdots q_n]^{\mathrm{T}} \in Q$ be a local coordinate of Q. Associated with q, we refer $\dot{q} = [\dot{q}_1 \cdots \dot{q}_n]^{\mathrm{T}} \in T_qQ$ as a tangent vector field.

A set of $n - m (m < n)$ differential equations:

$$A_i(t, q) + B_{i1}(q)\dot{q}_1 + \cdots + B_{in}(q)\dot{q}_n = 0, \quad i = 1, \ldots, n - m. \tag{1}$$

is called A-rheonomous affine constraints because the coefficient vector-valued function A depends on the time variable t explicitly. We now rewrite (1) as

$$A(t, q) + B(q)\dot{q} = 0, \tag{2}$$

where *a rheonomous affine term* $A(t, q) \in \mathbf{R}^{n-m}$ is a vector-valued function whose i-th entry is $A_i(t, q)$, and $B(q)$ is a matrix-valued function whose ij-th entry is $B_{ij}(q)$. It must be noted that this class of the A-rheonomous affine constraints (2) contains some important examples of mechanical systems as mentioned in Introduction. Now, we assume a sufficient condition on independency of the A-rheonomous affine constraints (2) as follows.

Assumption 1. The coefficient matrix $B(q)$ of the A-rheonomous affine constraints (2) has a row full-rank at any point $q \in Q$, that is,

$$\text{rank } B(q) = n - m, \quad \forall q \in Q \tag{3}$$

holds. □

Next, this subsection introduces a geometric representation method and provides some fundamental properties for the A-rheonomous affine constraints. From (3) in Assumption 1, the $n - m$ row vectors of $B(q)$ in the A-rheonomous affine constraints (2) are independent of each others at any point $q \in Q$. Hence, we here consider m vectors which are independent of each others and annihilators of the $n - m$ row vectors of $B(q)$, and denote them by $Y_1, \ldots, Y_m$ as vector fields on Q. In addition, we also denote a space spanned by $Y_1, \ldots, Y_m$, that is, a distribution on Q by

$$D := \text{span}\{Y_1, \cdots, Y_m\}. \tag{4}$$

Since the basial vectors of D: $Y_1, \ldots, Y_m$ are independent of each others at any point $q \in Q$, D is a nonsingular distribution, that is,

$$\dim D(q) = m, \quad \forall q \in Q \tag{5}$$

holds.

In order to represent the A-rheonomous affine constraints geometrically, we introduce an important vector fields on Q. A curve q: $I \to Q$ is said to be satisfied the A-rheonomous affine constraints (2) if for a vector field on Q: X and the generalized velocity of $q : \dot{q} \in T_{q(t)}Q$,

$$\dot{q}(t) - X(t, q(t)) \in D(q(t)), \quad \forall t \in I \tag{6}$$

holds as shown in Fig. 1. We call X *a rheonomous affine vector*. This definition is an extension of the one for the scleronomous affine constraints that do not contain the time variable.

Now, we show an essential property on the rheonomous affine vector X as follows.

Proposition 1. *For the A-rheonomous affine constraints (2), the component of the rheonomous affine vector field X, and a time interval $I \subset \mathbf{R}$,*

$$A(t, q) + B(q)X(t, q) = 0, \quad \forall t \in I, \quad \forall q \in Q \tag{7}$$

holds.

Proof. We assume that a velocity vector $\dot{q} \in T_qQ$ at a point $q \in Q$ satisfies the A-rheonomous affine constraints (2). Since $\dot{q} - X(t, q) \in D(q)$ holds, we have

$$\dot{q} - X(t, q) = \alpha_1(t, q)Y_1(q) + \cdots + \alpha_m(t, q)Y_m(q), \tag{8}$$

where $\alpha_1(t,q), \ldots, \alpha_m(t,q)$ are functions on Q. Now, multiply (8) by $B(q)$ from the left-hand side. Since the row vectors of $B(q)$ are annihilators of $Y_1(q), \ldots, Y_m(q)$, We then have

$$B(q)\{\dot{q} - X(t, q)\} = 0. \tag{9}$$

Furthermore, using (2), we can rewrite (9) as

$$B(q)\dot{q} - A(t, q) = 0 \tag{10}$$

and hence we obtain (7). □

Consequently, the A-rheonomous affine constraints can be geometrically represented as the following definition.

Definition 1. The A-rheonomous affine constraints (2) are geometrically represented by a pair (D,X), where D is a m-dimensional distribution defined by (4) and X is called *a rheonomous affine vector* and satisfies (7) (see Fig. 2). □

2.2. Complete nonholonomicity condition

Next, this subsection investigates complete nonholonomicity for the A-rheonomous affine constraints (2).

If all the $n - m$ A-rheonomous affine constraints (2) are nonintegrable, that is, there do not exist any independent first integrals of the time-varying affine constraints, then they are said to be *completely nonholonomic* or *completely nonintegrable*. Note that the Lie bracket [32–35] for two vector fields Z, W is defined as

$$[Z, W] := \frac{\partial W}{\partial q}Z - \frac{\partial Z}{\partial q}W. \tag{11}$$

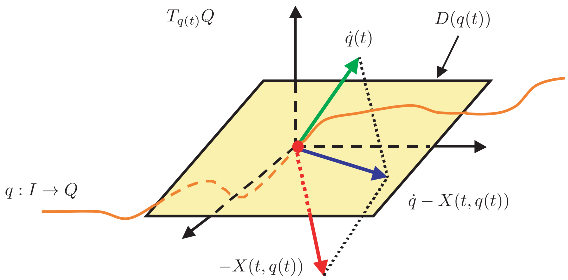

Fig. 2. Geometric representation of *A*-rheonomous affine constraints.

Now, in order to derive a nonintegrability condition for the *A*-rheonomous affine constraints, we first define a new bracket as follows.

Definition 2. For the vector fields defined on Q of the geometric representation of the *A*-rheonomous affine constraints (2): $X, Y_1, \ldots, Y_m$, *the rheonomous bracket* is an operator: $\langle\cdot,\cdot\rangle$: $TQ \times TQ \to TQ$ that satisfies the next three properties:

(a) For a rheonomous affine vector field X,

$$\langle X, X\rangle = 0 \tag{12}$$

holds.

(b) D_0 is defined as a set of vector fields that consists of $Y_1, \ldots, Y_m$ and iterated rheonomous brackets of $X, Y_1, \ldots, Y_m$ and does not contain X. For a rheonomous affine vector field X and a vector field $Z \in D_0$,

$$\begin{aligned} &\langle X, Z\rangle = \tfrac{\partial Z}{\partial t} + [X, Z], \quad Z \in D_0, \\ &\langle Z, X\rangle = -\tfrac{\partial Z}{\partial t} + [Z, X], \quad Z \in D_0 \end{aligned} \tag{13}$$

holds.

(c) For two vector fields Z, $W \in D_0$,

$$\begin{aligned} &\langle Z, Z\rangle := 0, \quad Z \in D_0, \\ &\langle Z, W\rangle := [Z, W], \quad Z, \quad W \in D_0 \end{aligned} \tag{14}$$

holds. □

It is the main characteristics of the rheonomous bracket that the rheonomous affine vector field X is perceived as special, and this yields an additional term of a time differential of a vector field as shown in the property (b). We note that from Definition 2 the rheonomous bracket is equivalent to the normal Lie bracket for scleronomous affine constraints, that is, constraints that do not contain the time variable explicitly. The rheonomous bracket will play important roles in derivation of a nonintegrability condition for the *A*-rheonomous affine constraints in this section and accessibility analysis in the next section. It turns out from the next proposition that the rheonomous bracket has the important characteristics in common with the normal Lie bracket.

Proposition 2. *For the vector fields on the geometric representation of the A-rheonomous affine constraints* (2): $X, Y_1, \ldots, Y_m$ *and the set of iterated vector fields of them:* D_0, *the following properties (a), (b), and (c) hold.*

(a) Bilinearlity:

$$\begin{aligned} &\langle X, aZ + bW\rangle = a\langle X, Z\rangle + b\langle X, W\rangle, \\ &\langle aZ + bW, X\rangle = a\langle Z, X\rangle + b\langle W, X\rangle, \quad Z,\ W \in D_0. \end{aligned} \tag{15}$$

(b) Skew-symmetry:

$$\langle X, Z\rangle = -\langle Z, X\rangle, \quad Z,\ W \in D_0. \tag{16}$$

(c) Jacobi's identity:

$$\langle\langle X, Z\rangle, W\rangle + \langle\langle Z, W\rangle, X\rangle + \langle\langle W, X\rangle, Z\rangle = 0, \quad Z,\ W \in D_0. \tag{17}$$

Proof. Based on the definition of the rheonomous bracket, we can calculate as follows:

$$\langle X, aZ+bW\rangle = \frac{\partial(aZ+bW)}{\partial t} + [X, aZ+bW] = a\frac{\partial Z}{\partial t} + a[X,Z] + b\frac{\partial W}{\partial t} + b[X,W] = a\langle X,Z\rangle + b\langle X,W\rangle,$$

$$\langle aZ+bW, X\rangle = -\frac{\partial(aZ+bW)}{\partial t} + [aZ+bW, X] = -a\frac{\partial Z}{\partial t} + a[Z,X] - b\frac{\partial W}{\partial t} + b[W,X] = a\langle Z,X\rangle + b\langle W,X\rangle.$$

Hence, we complete the proof of (a). Next, a simple calculation can show

$$\langle X,Z\rangle = \frac{\partial Z}{\partial t} + [X,Z] = -\left(-\frac{\partial Z}{\partial t} + [Z,X]\right) = -\langle Z,X\rangle.$$

Therefore, (b) holds. Finally, we shall prove (c). Since we can calculate the following:

$$\langle\langle X,Z\rangle, W\rangle = \left\langle \frac{\partial Z}{\partial t} + [X,Z], W\right\rangle = \left[\frac{\partial Z}{\partial t}, W\right] + [[X,Z],W] = \frac{\partial W}{\partial q}\frac{\partial Z}{\partial t} + \frac{\partial^2 Z}{\partial t\partial q}W + [[X,Z],W],$$

$$\begin{aligned}\langle\langle Z,W\rangle, X\rangle &= -\frac{\partial\langle Z,W\rangle}{\partial t} + [\langle Z,W\rangle, X] = -\frac{\partial [Z,W]}{\partial t} + [[Z,W],X] = -\frac{\partial}{\partial t}\left(\frac{\partial W}{\partial q}Z - \frac{\partial Z}{\partial q}W\right) + [[Z,W],X]\\ &= -\frac{\partial^2 W}{\partial t\partial q}Z - \frac{\partial W}{\partial q}\frac{\partial Z}{\partial t} + \frac{\partial^2 Z}{\partial t\partial q}W + \frac{\partial Z}{\partial q}\frac{\partial W}{\partial t} + [[Z,W],X],\end{aligned}$$

$$\langle\langle W,X\rangle, Z\rangle = [\langle W,X\rangle, Z] = \left[-\frac{\partial W}{\partial t} + [W,X], Z\right] = \left[-\frac{\partial W}{\partial t}, Z\right] + [[W,X],Z] = \frac{\partial Z}{\partial q}\frac{\partial W}{\partial t} + \frac{\partial^2 W}{\partial t\partial q}Z + [[W,X],Z],$$

we then obtain

$$\langle\langle X,Z\rangle, W\rangle + \langle\langle Z,W\rangle, X\rangle + \langle\langle W,X\rangle, Z\rangle = [[X,Z],W] + [[Z,W],X] + [[W,X],Z] = 0,$$

where we utilize Jacobi's identity for the normal Lie bracket. Consequently, the proof of (c) is completed. □

The properties in Proposition 2 can reduce the effort to calculate iterated rheonomous brackets in checking complete nonholonomicity of given A-rheonomous affine constraints. We now define a smallest and involutive rheonomous distribution $C_0(t,q)$ which contains $Y_1,\ldots,Y_m$ and satisfies $\langle X,W\rangle \in C_0$, $\forall W \in C_0$, that is, C_0 is spanned by all the rheonomous brackets of $X, Y_1,\ldots,Y_m$ with the exception of X. Then, we can derive a necessary and sufficient condition of complete nonholonomicity for the A-rheonomous affine constraints (2) as the next theorem.

Theorem 1. *For the A-rheonomous affine constraints (2), the following two statements are equivalent to each others. If they are satisfied, (2) are said to be completely nonholonomic.*

(a) *There exists no first integral of (2).*
(b) *For the rheonomous distribution C_0 and a time interval I,*

$$\dim C_0(t,q) = n, \quad \forall t \in I,\ \forall q \in Q \tag{18}$$

holds.

Proof. Consider the $(n+1)$-dimensional product space $\overline{Q} := \mathbf{R}\times Q$, where $\mathbf{R}$ is the space of the time variable t, and its local coordinate $\bar{q} := [t\, q^{\mathsf{T}}]^{\mathsf{T}}$. On $\overline{Q}$, the A-rheonomous affine constraints (2) are represented by Pfaffian equations of $n-m$ differential forms:

$$A(t,q)dt + B(q)dq = 0. \tag{19}$$

Since the rheonomous affine vector field X of the geometric representation satisfies (7), $m+1$ vector fields on $\overline{Q}$ which annihilate (19) are given by

$$\overline{X}(t,q) := \frac{\partial}{\partial t} \oplus X(t,q), \overline{Y}_i(q) := 0 \oplus Y_i(q), \quad i = 1,\ldots,m. \tag{20}$$

Now we define an involutive distribution $\overline{C}$ defined on $\overline{Q}$ which contains $\overline{X}, \overline{Y}_1,\ldots,\overline{Y}_m$ and iterated Lie brackets that consist of $\overline{X}, \overline{Y}_1,\ldots,\overline{Y}_m$. Therefore, a necessary and sufficient condition of complete nonintegrability for (19) is given by

$$\dim\overline{C}(t,q) = n+1, \quad \forall t \in I,\ \forall q \in Q. \tag{21}$$

(cf. Frobenius' theorem [32,33]). Calculating the iterated Lie brackets which consist of $\overline{X}, \overline{Y}_1, \ldots, \overline{Y}_m$, then we have

$$\begin{aligned}
&[\overline{X}(t,q), \overline{Y}_i(q)] = 0 \oplus \langle X(t,q), Y_i(q)\rangle,\\
&[\overline{X}(t,q), \overline{X}(t,q), \overline{Y}_i(q)]] = 0 \oplus \langle X(t,q), \langle X(t,q), Y_i(q)\rangle\rangle, \ldots\\
&[\overline{Y}_j(q), \overline{Y}_i(q)] = 0 \oplus \langle Y_j(q), Y_i(q)\rangle,\\
&[\overline{Y}_k(q), [\overline{Y}_j(q), \overline{Y}_i(q)]] = 0 \oplus \langle Y_k(q), \langle Y_j(q), Y_i(q)\rangle\rangle, \ldots
\end{aligned} \tag{22}$$

We can see that $\overline{X}$ is independent of $\overline{Y}_i, \ldots, \overline{Y}_m$ and the iterated Lie brackets (22). Then, the necessary and sufficient condition (21) is changed into the condition such that $\overline{Y}_1, \ldots, \overline{Y}_m$ and the iterated Lie brackets which consist of $\overline{X}, \overline{Y}_1, \ldots, \overline{Y}_m$ span an n-dimensional space. From (20) and (22), we can consider only $Y_1, \ldots, Y_m$ on Q instead of $\overline{Y}_1, \cdots, \overline{Y}_m$ on $\overline{Q}$, and iterated rheonomous brackets which consist of $X, Y_1, \ldots, Y_m$ on Q instead of Lie brackets which consist of $\overline{X}, \overline{Y}_1, \ldots, \overline{Y}_m$ on $\overline{Q}$. Therefore, a necessary and sufficient condition of complete nonholonomicity for the A-rheonomous affine constraints (2) is that $Y_1, \ldots, Y_m$ and the iterated rheonomous brackets which consist of $X, Y_1, \ldots, Y_m$ span an n-dimensional space, that is, (18) holds. □

From the result of Theorem 2, we can see that the complete nonholonomicity condition for the A-rheonomous affine constraints (2) has a similar structure as the one for the scleronomous affine constraints [23], and the rheonomous bracket introduced in Definition 2 plays a significant role.

3. Nonholonomic kinematic systems with *A*-rheonomous affine constraints

3.1. Mathematical model of NKSARAC

In this section, we develop a mathematical model of nonholonomic kinematic systems with A-rheonomous affine constraints (NKSARAC), which are a new class of kinematic control systems, and derive some characteristics of the NKSARAC.

First of all, this subsection derives the model of the NKSARAC. We first introduce a new classification of the generalized coordinate variable q. We divide q as $q = \left[q_a^\top\, q_b^\top\right]^\top$ with changing the order of q, where $q_a \in \mathbf{R}^r$ is called *an affine variable* and $q_b \in \mathbf{R}^{n-r}$ is called *a nonaffine variable*. By using these new variables, we rewrite the A-rheonomous affine constraints (2) as

$$A(t,q) + B_a(q)\dot{q}_a + B_b(q)\dot{q}_b = 0, \tag{23}$$

where $B_a \in \mathbf{R}^{(n-m)\times r}$, $B_b \in \mathbf{R}^{(n-m)\times(n-r)}$. Now, we give some assumptions on (23).

Assumption 2. For the A-rheonomous affine constraints (23), the following properties hold.

(a) For the number of the A-rheonomous affine constraints (23): $n-m$ and the dimension of the affine variable: r,

$$r \leqslant n-m \tag{24}$$

holds.
(b) The rheonomous affine term $A(t,q)$ depends on only the time variable t and the nonaffine variable q_b, that is, $A(t,q_b)$.
(c) B_a has a column full-rank.
(d) The equation which are obtained by substituting $\dot{q}_b = 0$ into the A-rheonomous affine constraints (23):

$$A(t,q_b) + B_a(q)\dot{q}_a = 0, \quad \forall t \in I \tag{25}$$

is physically satisfied. □

The condition (b) in Assumption 1 means that the effect caused by the rheonomous affine term for the system is determined by only the nonaffine variable, and excludes the complicated situation where the rheonomous affine term contains the affine variable. In addition, the condition (d) in Assumption 1 means that the variable which are changed by the effect caused by the rheonomous affine without external velocities and forces is the affine variable, and the generalized coordinate variable can be uniquely divided into the affine and nonaffine variables by physics consideration and (25). The rheonomous affine vector field X represents an effect of the A-rheonomous affine constraints to the system and the effect acts on only the affine variable. So, for the rheonomous affine vector field $\widehat{X} = \left[\widehat{X}_a^\top\, \widehat{X}_b^\top\right]^\top \in \mathbf{R}^n, \widehat{X}_a \in \mathbf{R}^r, \widehat{X}_b \in \mathbf{R}^{n-r}$,

$$A + B_a\widehat{X}_a = 0, \quad \widehat{X}_b = 0_{n-r} \tag{26}$$

must be satisfied, and hence it can be uniquely determined as

$$\widehat{X}(t,q) := \begin{bmatrix} -B_a(q)^\dagger A(t,q_b) \\ 0 \end{bmatrix}, \tag{27}$$

where $B_a^\dagger := (B_a^\top B_a)^{-1}B_a^\top$. Then, we consider control inputs to the system. We denote the control input by $u = [u_1 \cdots u_m]^\top \in \mathbf{R}^m$, and the directions which the control inputs act on is represented by $E(q)\dot{q}$ with the transformation matrix $E(q) \in \mathbf{R}^{m\times n}$. Now, we give assumptions on the control input as follows.

Assumption 3. For the control inputs $u = E(q)\dot{q}$, the following properties hold.

(a) The m control inputs are independent of each other, that is,

$$\operatorname{rank} E(q) = m, \quad \forall q \in Q \tag{28}$$

holds.

(b) The condition that the constrained direction of the A-rheonomous affine constraints (23) and the direction of the control inputs do not interfere with each other, that is,

$$\operatorname{rank}\begin{bmatrix} B(q) \\ E(q) \end{bmatrix} = n, \quad \forall q \in Q \tag{29}$$

holds. □

We then derive the NKSARAC. The drift vector of the NKSARAC is equal to the rheonomous affine vector field $\widehat{X}(t,q)$ defined by (27). From the condition (b) in Assumption 3, we set the vector fields as

$$\widehat{Y}_i(q) := \begin{bmatrix} B(q) \\ E(q) \end{bmatrix}^{-1} \begin{bmatrix} 0_{n-m} \\ e_i \end{bmatrix}, \quad i = 1, \ldots, m, \tag{30}$$

where $e_i \in \mathbf{R}^m$ is a unit vector whose i-th entry is *1* and the others are all *0*. We also use the notation:

$$\widehat{Y}(q) := [\widehat{Y}_1 \cdots \widehat{Y}_m] = \begin{bmatrix} \widehat{Y}_a(q) \\ \widehat{Y}_b(q) \end{bmatrix}, \tag{31}$$

$\widehat{Y}_a \in \mathbf{R}^{r\times m}$, $\widehat{Y}_b \in \mathbf{R}^{(n-r)\times m}$. Consequently, *the nonholonomic kinematic system with A-rheonomous affine constraints (NKSARAC)* is obtained by

$$\dot{q} = \widehat{X}(t,q) + \sum_{i=1}^{m} \widehat{Y}_i(q) u_i. \tag{32}$$

The first term of the right-hand side of (32) represents the effect of the rheonomous affine term to the system, and the second term of the right-hand side of (32) indicates the effect of the control inputs. The NKSARAC (32) is formulated as a time-varying nonlinear asymmetric affine control system, which has a non-zero drift term. On the other hand, For linear constraints, i.e., $A(t,q) \equiv 0$, the kinematic system derived from them is represented by a time-invariant nonlinear symmetric control system. Compared to the system, the NKSARAC is more complicated and difficult to analyze because it has a non-zero and time-varying drift term.

3.2. Linearly-approximated system of NKSARAC

Next, in this subsection, we calculate a linearly-approximated system of the NKSARAC (32). In general, a linearly-approximated system of an original nonlinear system is essential in nonlinear control theory, because it tells us the local properties of the original system.

The set of the equilibrium points for the NKSARAC (32) is given by

$$U^e := \left\{ q = [q_a^\top \, q_b^\top]^\top \in Q \,|\, A(t, q_b) = 0, \quad \forall t \in I \right\}. \tag{33}$$

It is noted that there exist not only equilibrium points but also nonequilibrium points for the NKSARAC because of the existence of the drift term $\widehat{X}$ in (32). The linearly-approximated system of the NKSARAC is given by the following theorem.

Theorem 2. *The linearly-approximated system of the NKSARAC (32) at an equilibrium point $q^e = [q_a^{e\top} \, q_b^{e\top}]^\top \in U^e$ is given by*

$$\dot{q} = \underbrace{\begin{bmatrix} O_r & -B_a^{\dagger}(q^e)\frac{\partial A}{\partial q_b}(t, q_b^e) \\ O_{n-r,r} & O_{n-r} \end{bmatrix}}_{\mathcal{A}} (q - q^e) + \underbrace{\begin{bmatrix} \widehat{Y}_a(q^e) \\ \widehat{Y}_b(q^e) \end{bmatrix}}_{\mathcal{B}} u. \tag{34}$$

Proof. The coefficient matrices of the linearly-approximated system can be calculated by using the next definition.

$$\mathcal{A} = \left.\frac{\partial \widehat{X}}{\partial q}\right|_{q=q^e}, \quad \mathcal{B} = \left.\widehat{Y}\right|_{q=q^e}. \tag{35}$$

The details of the derivation are omitted. □

3.3. Accessibility analysis on NKSARAC

Finally, we discusses accessibility of the NKSARAC in this subsection. Accessibility is one of the important and fundamental properties for nonlinear control systems.

Now, we sum up some concepts of accessibility of nonlinear control systems [32,33,36]. Consider a general nonlinear affine control system:

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \tag{36}$$

where $x = [x_1 \cdots x_n] \in \mathbf{R}^n$ is a state variable, $u \in [u_1 \cdots u_m] \in \mathbf{R}^m$, and $f, g_1, \ldots, g_m$ are vector fields. For a given point x_0, let $\Lambda^V(t, x_0)$ be the set of points x that there exists a neighborhood V of x_0 and an admissible control u such that there is a trajectory $x(\tau)$ of (32) which satisfies $x(\tau) \in V (0 \leqslant \tau \leqslant t)$ and $x(0) = x_0$, $x(t) = x$ (see Fig. 3). This set is called *the accessible set from x_0 at time t*. Let $\Lambda_t^V(x_0)$ be the other set defined by a sum of $\Lambda^V(\tau, x_0)$ from time 0 to t, that is,

$$\Lambda_t^V(x_0) := \bigcup_{\tau=0}^{t} \Lambda^V(\tau, x_0). \tag{37}$$

This set is called *the accessible set from x_0 in up to time t*. In addition, we also define *the accessible set from x_0* as

$$\Lambda^V(x_0) = \lim_{t\to\infty} \Lambda_t^V(x_0). \tag{38}$$

If $\Lambda_t^V(x_0)$ contains a non-empty open set of the configuration for all neighborhoods V of x_0, then the system is called *locally accessible from x_0*. Moreover, if for any neighborhood V of x_0, $\Lambda^V(t, x_0)$ contains a non-empty open set for any sufficiently small $t > 0$, then the system is called *strongly locally accessible from x_0*. Now, we prove the following theorem on accessibility of the NKSARAC (32).

Theorem 3. *The NKSARAC (32) is strongly accessible if and only if the A-rheonomous affine constraints (23) are completely nonholonomic.*

Proof. First, we consider *the extended NKSARAC* on the product manifold $\overline{Q} = \mathbf{R} \times Q$ with its local coordinate $\bar{q} := [t\, q^\top]^\top$:

$$\dot{\bar{q}} = \underbrace{\begin{bmatrix} 1 \\ \widehat{X}(\bar{q}) \end{bmatrix}}_{\overline{X}(\bar{q})} + \sum_{i=1}^{m} \underbrace{\begin{bmatrix} 0 \\ \widehat{Y}_i(q) \end{bmatrix}}_{\overline{Y}_i(q)} u_i, \tag{39}$$

where

$$\overline{X}(\bar{q}) := \frac{\partial}{\partial t} \oplus X(\bar{q}), \quad \overline{Y}_i(q) := 0 \oplus Y_i(q), \quad i = 1, \ldots, m. \tag{40}$$

It must be noted that in order to derive (39), we use the constraint on the time variable $t : \dot{t} = 1$. We here define a smallest and involutive distribution on $\overline{Q}$: $\overline{C}$ which contains $\overline{X}, \overline{Y}_1, \ldots, \overline{Y}_m$ and satisfies $[\overline{Z}, \overline{W}] \in \overline{C}, \forall \overline{Z}, \overline{W} \in \overline{C}$. A necessary and sufficient condition so that the accessible set $\Lambda^{\overline{V}}(\bar{q}_0)$ of the extended NKSARAC (39) has a non-empty open set in $\overline{V} \subset \overline{Q}$, that is, local accessibility is

$$\dim \overline{C}(\bar{q}) = n + 1, \quad \forall \bar{q} \in \overline{Q}. \tag{41}$$

Note that local accessibility for the extended NKSARAC (39) is equivalent to strong local accessible for the NKSARAC (32), i.e., the accessible set $\Lambda_t^V(q_0)$ of the NKSARAC (39) has a non-empty open set in $V \subset Q$ for any sufficiently small $t > 0$. Calculating iterated Lie brackets of $\overline{X}, \overline{Y}_1, \ldots, \overline{Y}_m$, we obtain

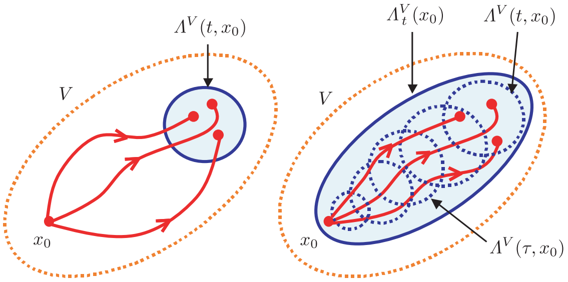

Fig. 3. Accessible sets.

$$\begin{aligned}
&[\overline{X}(\bar{q}), \overline{Y}_i(q)] = 0 \oplus \langle \widehat{X}(\bar{q}), \widehat{Y}_i(q) \rangle, \\
&[\overline{X}(\bar{q}), \overline{X}(\bar{q}), \overline{Y}_i(q)]] = 0 \oplus \langle \widehat{X}(\bar{q}), \langle \widehat{X}(\bar{q}), \widehat{Y}_i(q) \rangle \rangle, \ldots \\
&[\overline{Y}_j(q), \overline{Y}_i(q)] = 0 \oplus \langle \widehat{Y}_j(q), \widehat{Y}_i(q) \rangle, \\
&[\overline{Y}_k(q), [\overline{Y}_j(q), \overline{Y}_i(q)]] = 0 \oplus \langle \widehat{Y}_k(q), \langle \widehat{Y}_j(q), \widehat{Y}_i(q) \rangle \rangle, \ldots
\end{aligned} \tag{42}$$

Similar to the proof of Theorem 1, $\overline{X}$ is independent of $\overline{Y}_i, \cdots, \overline{Y}_m$ and the iterated Lie brackets (22). So, the necessary and sufficient condition (21) is changed into the condition such that $\overline{Y}_1, \ldots, \overline{Y}_m$ and the iterated Lie brackets which consist of $\overline{X}, \overline{Y}_1, \ldots, \overline{Y}_m$ span an n-dimensional space. In addition, we only have to consider $\widehat{Y}_1, \ldots, \widehat{Y}_m$ on Q instead of $\overline{Y}_1, \ldots, \overline{Y}_m$ on $\overline{Q}$, and iterated rheonomous brackets which consist of $X, Y_1, \ldots, Y_m$ on Q instead of Lie brackets which consist of $\overline{X}, \overline{Y}_1, \ldots, \overline{Y}_m$ on $\overline{Q}$. Hence, a necessary and sufficient condition of local accessibility for the extended NKSARAC (39) is that $\widehat{Y}_1, \ldots, \widehat{Y}_m$ and the iterated rheonomous brackets which consist of $\widehat{X}, \widehat{Y}_1, \ldots, \widehat{Y}_m$ span an n-dimensional space. This condition is equivalent to

$$\dim C_0(t, q) = n, \quad \forall t \in I, \ \forall q \in Q \tag{43}$$

for a distribution C_0 which is defined in Section 2.2 That is to say, this means the A-rheonomous affine constraints (23) is completely nonholonomic. Consequently, this theorem is proven. □

The main merit of Theorem 3 is that we can check strong accessibility of the NKSARAC (32) by the result of the complete nonintegrability test for the A-rheonomous affine constraints (32) shown in Theorem 1. Hence, this result saves us a lot of effort of calculating iterated Lie and rheonomous brackets for vector fields. In addition, the results in Theorem 1 can be interpreted as a natural extension of the case for the scleronomous affine constraints [23,24,26]

4. Examples

4.1. Boat on running river with time-varying stream

In this section, we show two types of physical example in order to confirm the results. First, we consider *a boat on a running river with a time-varying stream* as shown in Fig. 4. Set the x-axis and y-axis to the transverse direction and the downstream direction of the river, respectively, and denote the center of inertia of the boat by (x, y). In addition, let θ be the angle of the boat. Let $V(t, x)$ be a stream of the river that depends on the time variable t as well as the transverse position x, that is, the stream changes as time goes by. It is assumed that the boat is affected by the stream to the downstream direction of the river according to the angle of the boat θ, and hence the boat drifts to the y-direction. So, we can see that the affine variable of the system is y, that is, $q_a = y \in \mathbf{R}^1$ with $r = 1$ and the nonaffine variable is given by $q_b = [x\theta]^\top \in \mathbf{R}^2$. Then, the generalized coordinate of this system is represented by $q = \left[q_a \ q_b^\top\right]^\top \in \mathbf{R}^3$ with $n = 3$.

Considering the balance of the velocities in both the heading and side directions of the boat, we have the A-rheonomous affine constraints of this system as

$$\underbrace{V(t,x)\cos^3\theta}_{A(t,q)} + \underbrace{[\sin\theta \ -\cos\theta \, 0]}_{B(q)} \begin{bmatrix} \dot{y} \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = 0, \tag{44}$$

where $m = 2$. We assume that the velocity of the traveling direction of the boat and the angular velocity the angle of the boat can be controlled. So, the control inputs $u = [u_1 u_2]^\top \in \mathbf{R}^2$ are defined by the transformation matrix:

$$E(q) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{45}$$

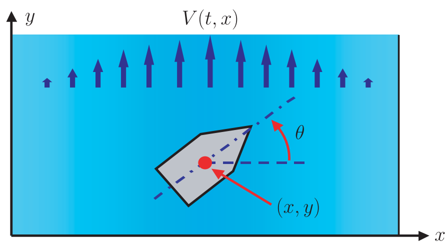

Fig. 4. A boat on a running river with a time-varying stream.

Hence, from (32), the NKSARAC of this system is represented by

$$\begin{bmatrix} \dot{y} \\ \dot{x} \\ \dot{\theta} \end{bmatrix} = \underbrace{\begin{bmatrix} V(t,x)\cos^2\theta \\ 0 \\ 0 \end{bmatrix}}_{\widehat{X}} + \underbrace{\begin{bmatrix} \sin\theta \\ \cos\theta \\ 0 \end{bmatrix}}_{\widehat{Y}_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\widehat{Y}_2} u_2. \tag{46}$$

For the vector fields $\widehat{X}, \widehat{Y}_1, \widehat{Y}_2$, calculate some iterated rheonomous brackets:

$$\begin{aligned} \langle \widehat{X}, \widehat{Y}_1 \rangle &= \frac{\partial \widehat{Y}_1}{\partial t} + [\widehat{X}, \widehat{Y}_1] = \begin{bmatrix} -\frac{\partial V}{\partial x}\sin\theta\cos^2\theta \\ 0 \\ 0 \end{bmatrix}, \\ \langle \widehat{X}, \widehat{Y}_2 \rangle &= \frac{\partial \widehat{Y}_2}{\partial t} + [\widehat{X}, \widehat{Y}_2] = \begin{bmatrix} 2V(t,x)\sin\theta\cos\theta \\ 0 \\ 0 \end{bmatrix}, \\ \langle \widehat{Y}_1, \widehat{Y}_2 \rangle &= [\widehat{Y}_1, \widehat{Y}_2] = \begin{bmatrix} -\cos\theta \\ \sin\theta \\ 0 \end{bmatrix}. \end{aligned} \tag{47}$$

Consequently, we can see that

$$C_0 = \mathrm{span}\{\widehat{Y}_1, \widehat{Y}_2, \langle \widehat{Y}_1, \widehat{Y}_2 \rangle\} \tag{48}$$

and hence

$$\dim C_0(t,q) = 3 = n, \quad \forall t \in \mathbf{R}, \ \forall q \in Q \tag{49}$$

holds. So, the A-rheonomous affine constraints of this system (44) are completely nonholonomic from Theorem 1. As a result, from Theorem 2, the NKSARAC of the boat on a running river (46) is strongly accessible at any point $q \in Q$.

4.2. Ball on rotating table with time-varying angular velocity

We next consider another example, *a ball on a rotating table with a time-varying angular velocity*. We deal with an undeformable ball and a rotating table that turns at a time-varying angular velocity as shown in Fig. 5. We assume that the ball does not slip and rotates with a velocity received by the rotating table. Consider the $x - y$ coordinate system so that the origin of it: O is coincident with the center of the rotating table, and let (x, y) be the point with which the ball contacts. We denote the angles of rotation of the ball by $(\theta_1, \theta_2, \theta_3)$ as rotational angles of the x, y and z axes, respectively. Since the ball rotates by the effect of the velocities given by the rotating table, the affine variable of the system is θ_1 and θ_2, that is, $q_a = [\theta_1\, \theta_2]^{\mathsf{T}} \in \mathbf{R}^2$ with $r = 2$ and the nonaffine variable is given by $q_b = [x\ y\ \theta_3]^{\mathsf{T}} \in \mathbf{R}^3$. Hence, the generalized coordinate of this system is represented by $q = [q_a^{\mathsf{T}}\ q_b^{\mathsf{T}}]^{\mathsf{T}} \in \mathbf{R}^5$ with $n = 5$. We also use the parameters of the system; R: the radius of the ball, $\Omega(t) > 0$: the time-varying angular velocity of the rotating table.

Considering the balance of the velocities in both x and y directions of the ball, we have the A-rheonomous affine constraints of this system as

$$\underbrace{\begin{bmatrix} \Omega(t)y \\ -\Omega(t)x \end{bmatrix}}_{A(t,q)} + \underbrace{\begin{bmatrix} 0 & -R & 1 & 0 & 0 \\ R & 0 & 0 & 1 & 0 \end{bmatrix}}_{B(q)} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{x} \\ \dot{y} \\ \dot{\theta}_3 \end{bmatrix} = 0, \tag{50}$$

where $m = 3$. It is assumed that the angular velocity of θ_1, θ_2, and θ_3 directions can be controlled. So, the control inputs $u = [u_1\, u_2\, u_3]^{\mathsf{T}} \in \mathbf{R}^2$ are defined by using the transformation matrix:

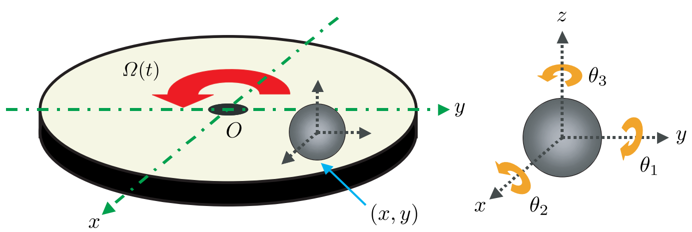

Fig. 5. A ball on a rotating table with a time-varying angular velocity.

$$E(q) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{51}$$

Hence, from (32), the NKSARAC of this system is represented by

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \dot{x} \\ \dot{y} \\ \dot{\theta}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\Omega(t)x}{R} \\ \frac{\Omega(t)y}{R} \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\widehat{X}} + \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -R \\ 0 \end{bmatrix}}_{\widehat{Y}_1} u_1 + \underbrace{\begin{bmatrix} 0 \\ 1 \\ R \\ 0 \\ 0 \end{bmatrix}}_{\widehat{Y}_2} u_2 + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\widehat{Y}_3} u_3. \tag{52}$$

For the vector fields $\widehat{X}, \widehat{Y}_1, \widehat{Y}_2, \widehat{Y}_3$, calculate some iterated rheonomous brackets:

$$\begin{aligned}
\langle \widehat{X}, \widehat{Y}_1 \rangle &= \frac{\partial \widehat{Y}_1}{\partial t} + [\widehat{X}, \widehat{Y}_1] = \begin{bmatrix} 0 \\ \Omega(t) \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
\langle \widehat{X}, \widehat{Y}_2 \rangle &= \frac{\partial \widehat{Y}_2}{\partial t} + [\widehat{X}, \widehat{Y}_2] = \begin{bmatrix} -\Omega(t) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
\langle \widehat{X}, \widehat{Y}_3 \rangle &= \frac{\partial \widehat{Y}_3}{\partial t} + [\widehat{X}, \widehat{Y}_3] = 0, \\
\langle \widehat{X}, \langle \widehat{X}, \widehat{Y}_1 \rangle \rangle &= \frac{\partial \langle \widehat{X}, \widehat{Y}_1 \rangle}{\partial t} + [\widehat{X}, \langle \widehat{X}, \widehat{Y}_1 \rangle] = \begin{bmatrix} 0 \\ \frac{d\Omega(t)}{dt} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
\langle \widehat{X}, \langle \widehat{X}, \widehat{Y}_2 \rangle \rangle &= \frac{\partial \langle \widehat{X}, \widehat{Y}_2 \rangle}{\partial t} + [\widehat{X}, \langle \widehat{X}, \widehat{Y}_2 \rangle] = \begin{bmatrix} -\frac{d\Omega(t)}{dt} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \\
&\vdots \\
\langle \widehat{Y}_1, \widehat{Y}_2 \rangle &= \langle \widehat{Y}_2, \widehat{Y}_3 \rangle = \langle \widehat{Y}_3, \widehat{Y}_1 \rangle = 0.
\end{aligned} \tag{53}$$

Then, we have

$$C_0 = \operatorname{span}\{\widehat{Y}_1, \widehat{Y}_2, \widehat{Y}_3, \langle \widehat{X}, \widehat{Y}_1 \rangle, \langle \widehat{X}, \widehat{Y}_2 \rangle\} \tag{54}$$

and therefore

$$\dim C_0(t, q) = 5 = n, \quad \forall t \in I := \mathbf{R}, \ \forall q \in Q \tag{55}$$

holds. Hence, the A-rheonomous affine constraints of this system (50) are completely nonholonomic from Theorem 1. Consequently, we can see that the NKSARAC of a ball on a rotating table (52) is strongly accessible at any point $q \in Q$ from Theorem 2.

5. Conclusions

In this paper, we have modeled and analyzed the nonholonomic kinematic systems with A-rheonomous affine constraints (NKSARAC). We first have obtained a necessary and sufficient condition of complete nonholonomicity for the A-rheonomous affine constraints. Next, we have formulated the NKSARAC from the A-rheonomous affine constraints and the control inputs under some assumptions. Then, a necessary and sufficient condition of strong accessibility for the NKSARAC has been derived. Throughout this paper, we have introduced and developed a new class of nonholonomic control systems.

Our future work includes the following problems: (i) controllability and stabilizability analysis on the NKSARAC, (ii) controller synthesis for the NKSARAC, (iii) applications to various mechanical systems, (iii) extensions to more general classes of constraints, for example, *fully rheonomous affine constraints*: $A(t,q) + B(t,q)\dot{q} = 0$.

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